

Uplifting The Quality Of Exploratory Factor Analysis: Insights from Mathematics

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ABSTRACT

While Common Factor Analysis (FA) has been recommended by literature over Principal Components Analysis (PCA), the logic for the recommendation is often based on intuitive understanding of the matrices (R_{fa} vs. R_{pca}) used by these factor analysis models. While PCA uses correlation matrix $R = R_{pca} = X^T X$ where 1's are on the diagonal, FA uses R_{fa} where 1's on the diagonal of R are replaced by commonalities (h^2) of observed variables. The intuitive belief is that eigenvalue-eigenvector procedure on the matrix R_{fa} will deliver trustworthy results because there is no unique variance u^2 in R_{fa} . This paper will show mathematically that this belief is false. The paper will recommend that FA and PCA should be used together to extract factors from empirical data.

Keywords: Eigenvalue-eigenvector procedure, PCA, FA, EFA Applications; Multivariate Data Interpretation.

1. INTRODUCTION

After ground-breaking efforts by Pearson (1901) and Spearman (1904, 1927) in designing and using factor analysis in psychological research, factor analysis has recently become widely used in marketing studies and been one of the major contributors to the progression of marketing research. A great variety of factor analytical models have been developed to suit differentiated marketing research needs. Among these models, Principal Components Analysis (PCA) and Common Factor Analysis (FA) are probably the most used EFA models. When using EFA, researchers often face choices concerning (1) factor model; (2) number of factors; (3) rotation; (4) factor interpretation (Conway and Huffcutt, 2003; Fabrigar, Wegener, MacCallum and Strahan, 1999; Ford, MacCallum and Tait, 1986; Hair et al., 2006; Lattin et al., 2003; and Tabachnick and Fidell, 2001). At each choice point, confusions arise due to differences and similarities between choices they have to make. Their confusions are also in regard to exactly what each technique actually does. Verbal logics are often used in literature to persuade users to use one choice against another. Take "(3) rotation" for example, Lattin et al. (2003:90) suggest, "... that the interpretation of each principal component is easier when the loadings take on values that are either large (in absolute value) or near zero. To facilitate interpretation, it is possible to rotate the retained components (i.e. change their orientation) so that the loading matrix takes on a simpler structure." In fact, the above suggestion makes sense in practice. However, it does not make sense if you are skeptical of it. For example, if the retained factors have to be rotated, why bother to retain them in the first place? In other words, could we just vision a factor arbitrarily and then rotate it for the same purpose? (2) When a retained factor is rotated, will the rotated factor necessarily be the same factor? Although experienced researchers suggested that they are not the same, there seems no literature confirm that explicitly. This paper explores these issues and aims to provide a resolution.

2. LITERATURE REVIEW OF EFA

As a general knowledge, "If specific experiments or behavior characteristics of a population or sample are regarded as measures or variables, factor analysis can be described as a broad category of approaches to determining the structure of relations among measures or variables." (Nunnally and Bernstein, 1994:447) Therefore, EFA may be used to suggest:

1. groupings or clusterings of variables,
2. which variables belong to which group and how strongly they belong,
3. how many dimensions are needed to explain the relations among the variables,
4. a frame of reference to describe the relations among the variables more conveniently,
5. scores of individual on such groupings.

Gorsuch (1983) calls EFA a logical extension of methods people always used to understand their world. So, EFA is used extensively to suggest or discover strong correlations among variables. (Nunnally and Bernstein, 1994) It has been suggested that EFA is compulsory if there are no *a priori* hypotheses for the data to verify (Hurley, Scandura, Schriesheim, Brannick, Seers, Vandenberg and William, 1997).

In regard to the quality of EFA application, Ford et al. (1986), Fabrigar et al. (1999), Conway and Huffcutt (2003) identified four issues: (1) the choice of factor model to use; (2) the decision about the number of factors to retain; (3) the methods of rotation; (4) the interpretation of the factor solution. After reviewing more than 500 journal papers using EFA in three major psychology journals over a combination of 25-year period, they concluded that EFA was often poorly applied in empirical studies. They noticed that users tended to favor (a) components analysis; (b) extracting factors with eigenvalues greater than one; (c) rotating the solution orthogonally; (d) interpreting loadings above a minimum value. In general, the findings of these factor analysis applications were distorted and potentially meaningless solutions.

However, by comparing reviews of Ford et al.'s (1986), Fabrigar et al.'s (1999), and Conway and Huffcutt (2003), Conway and Huffcutt noticed some improvements on the decision of rotation (more oblique rotations were reported in recent studies) and the number of factor decision (more use of multiple criteria). Regarding to the decisions on factor model, they found no improvement at all. Moreover, Conway and Huffcutt (2003) also noticed that the quality of factor analysis highly correlated with research design of factor applications.

Finally, common factor analysis, a combination of techniques for factor retention and oblique rotation were recommended for EFA applications on both theoretical ground and empirical evidence. In combination, they can always deliver better factor solutions than otherwise. (Ford et al., 1986; Gorsuch, 1990, Fabrigar et al., 1997 and Convey and Huffcutt, 2003)

Figure 1 shows the traditional interpretation on factor analysis graphically and demonstrates the differences between principal components analysis (PCA) and common factor analysis (FA). Table 1 summarizes differences and similarities between the two models along 13 model attributes.

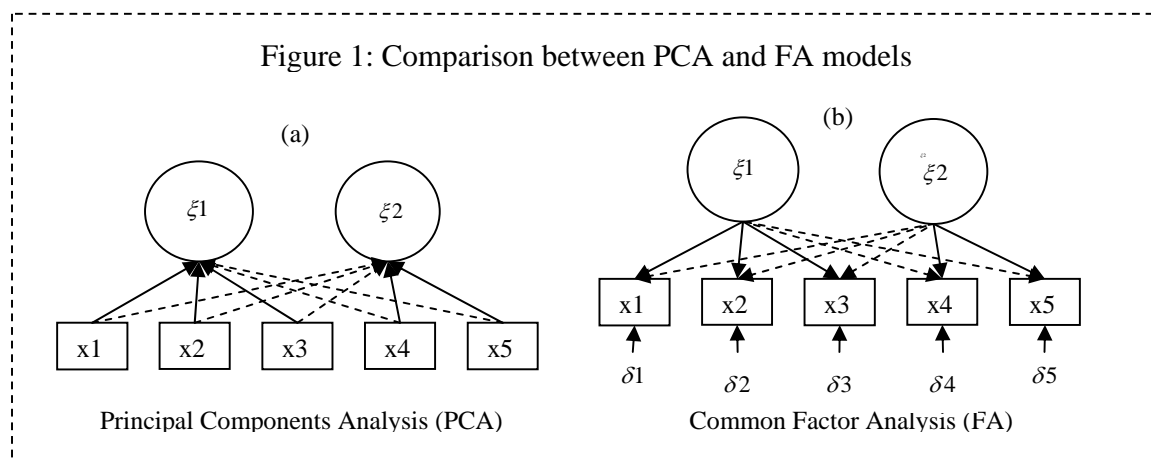


Table 1: comparison between PCA and EFA

Attribute	PCA	FA
Goals	Summarization and exploratory: <ul style="list-style-type: none"> Result in convenient grouping of variables Maximize the ability to explain variance. Pragmatic (data reduction). 	Exploratory: <ul style="list-style-type: none"> To understand latent constructs and generate hypothesis. Theoretical.
Measurement theory	<ul style="list-style-type: none"> No measurement model assumed. The relationship between variables and components are not specified. PCA is often assumed to be a special case for FA because e_j is assumed to be zero (0). 	<ul style="list-style-type: none"> Measurement model is assumed implicitly. The relationship between variables and latent factors are not specified. Assuming underlying dimensionality of data = latent factors.
Mathematical equation	PCs are taken to be linear combination of x_1, x_2, \dots, x_p . $z = u_1x_1 + u_2x_2 + \dots + u_px_p$, or, $Z = XU$, in matrix form.	Variables are assumed to be linear combination of the factors plus an error term (or specific factors) for each variable. $x_j = \sum_{k=1}^m \lambda_{jk} \xi_k + e_j$, or $X = \Xi \Lambda' + \Delta$, in matrix form.
Mathematical procedure for factor analysis	Eigenvalue-eigenvector solution procedure with 1's on the diagonal of the R matrix	Eigenvalue-eigenvector solution procedure with communalities ($1 - \Psi_j$, Ψ_j is the error variance of the variable x_j) replacing 1's on the diagonal of the R matrix.
Variance	<ul style="list-style-type: none"> All the variance in the input variables is analyzed. Some error variance were identified and discarded by factor analysis procedure. Identified components still contain some, but reduced error variance. 	<ul style="list-style-type: none"> It is assumed that only shared variance is analyzed. Error variance is believed to be identified and eliminated in the input variables. Identified factors are believed to be error free.
Relationship between variables and factors	Non-causal: Components are produced by aggregates of correlated measurement variables.	Causal: Variance of measurement variables are believed to be caused by latent factors in theory.
Factors (or components)	<ul style="list-style-type: none"> Components were derived from statistical results. Researcher determines number of components to retain, according to interpretability and scientific utility. All components are treated the same. To retain sufficient components to achieve a high percentage of the original variation. 	<ul style="list-style-type: none"> Factors were derived from statistical results. Researcher determines number of factors to retain, according to pre-perceived theory and scientific utility. All factors are treated the same. High percentage of the original variation and communalities play a role in determine the no. of factors to extract. In practice, no. of PCs derived from factor-analyzing R matrix are often been used as no. of factors in the

		subsequent FA with corresponding R matrix.
Loading matrix	<ul style="list-style-type: none"> All variables are related to every component through factor loading matrix. The analytic technique assigns variables to components. 	<ul style="list-style-type: none"> All variables are related to every factor by factor loading matrix. The analytic technique assigns variables to factors.
Rotation	<ul style="list-style-type: none"> To find a more interpretable simple structure solution. Rotation choice: orthogonal or oblique. 	<ul style="list-style-type: none"> To find a more interpretable simple structure solution. Rotation choice: orthogonal or oblique.
Capitalize on chance factors	A reduced portion of the original error still exists in factor solution.	Less likely, the solution contains error. But we have no way to know it.
Indicators of dimensionality	<ul style="list-style-type: none"> Eigenvalue-greater-than-1, Scree-plot test, Parallel analysis, <i>A priori</i> number of factors, Parsimony, High proportion of variance, Minimum average partial, A combination of techniques, etc. 	<ul style="list-style-type: none"> Eigenvalue-greater-than-0, Scree test, Parallel analysis, <i>A priori</i> number of factors, Parsimony, <i>Chi-square test (maximum likelihood)</i> High proportion of variance, A combination of techniques, The most interpretable solution, etc.
Application	<ul style="list-style-type: none"> Scale development. Evaluating construct validity. Hypothesis testing. 	<ul style="list-style-type: none"> Scale development. Evaluating construct validity. Hypothesis testing.
Sample size	<ul style="list-style-type: none"> Sample/variable ratio: at least 5:1; bigger than 10:1 preferred. Variable/factor ratio: 4:1. 	<ul style="list-style-type: none"> Sample/variable ratio: at least 5:1; bigger than 10:1 preferred. Variable/factor ratio: 4:1.

Following sources are used heavily for the creation of the above table: Lattin et al. (2003); Tabachnick and Fidell (2001); Hair et al. (2006); Ford et al. (1986); Hurley et al. (1997); Fabrigar et al. (1999); Conway and Huffcutt (2003).

In regard to mathematical capability of factor analysis, Sorbon (1982) said, the terms 'factors or constructs' will never be anything more than is contained in the observed variables and will never be anything beyond what has been specified approximations in the model. Thus, factors (constructs or unobservable variables) are merely the product of a model specification and used as facilities in specifying structural models. The above remarks should be taken as (1) a warning that the use of factor analysis is not a panacea for every human problem; (2) a dangerous act if EFA is used only for its abstract mathematical procedures or computer programs. Only clearly defined problems can be solved effectively using EFA. Therefore, as Nunnally and Bernstein (1994) points out, the conceptual and mathematical models of factor analysis should be both carefully thought out and interwoven in better factor analysis investigations. Toward that purpose, one must first distinguish between them to have a better understanding of factor analysis to apply it in research reality. In the following discussion, our focus is only on analyzing mathematical understanding of EFA and how that understanding could be used to improve the quality of EFA applications.

3. ANALYSIS

For EFA, there are two solution approaches: (1) Common Factor Analysis, which tries to separate the common factor variance that each variable can contribute to factors from the variance that is unique in itself; (2) Principal Components Analysis, which are simply linear combinations of observables and therefore observables in their own right. The essential point common to all EFA is that all seek to explain as much about the variables as possible with the fewest factors. (Nunnally and Bernstein, 1994)

EFA usually begins by computing a correlation matrix R . Before discussing EFA in details, we need to mention that PCA and FA share the same eigenvalue-eigenvector procedure, but the matrices they use are different. While the symmetric matrix R_{pca} PCA use has 1's on the diagonal and therefore, is positive definite, the symmetric matrix R_{fa} FA use has the commonalities of variables ($1 - \Psi_j < 1$) on the diagonal and its quadratic form will be indefinite. They both seek smaller factor dimensions for the purpose of parsimony. Since symmetric matrices are used in both cases ($(A^T A)^T = A^T A$), the resulting factor axes (eigenvectors) will be orthogonal to each other.

3.1 Principal Components Analysis (PCA):

To begin, let's assume that we have p survey variables and wish to obtain m ($m \ll p$) factors. Since it is almost impossible for any two survey variables having identical data set under survey conditions, the p observed variables can be regarded as independent. What PCA does in factor analysis is to find a full set of new p axes in the data space created by survey variables. As a result of PCA, a few of new dimensions are maximized with larger variances and the rest dimensions are minimized with reduced variances. So is the statement of retaining a greater proportion of original variation by a much smaller number of derived or underlying variables (factors).

Since PCA regards all variances in the survey data are systematic, the derived factors are simply linear combinations of observables. The derived variables z_1, z_2, \dots, z_p , are taken to be linear functions of x_1, x_2, \dots, x_p with the following properties:

- a) z_1 has maximum possible variance among all possible linear functions:
 $z = a_1 x_1 + a_2 x_2 + \dots + a_p x_p$, of x_1, x_2, \dots, x_p .
- b) z_2 has maximum possible variance among all possible linear functions of x_1, x_2, \dots, x_p , subject to z_2 being uncorrelated with z_1 .
- c) In general, z_k has maximum possible variance among all possible linear functions of x_1, x_2, \dots, x_p , subject to z_k being uncorrelated with z_1, z_2, \dots, z_{k-1} , for $2 \leq k \leq p$.

Or $Z = AX$, in matrix form

If $z_1, z_2, \dots, z_{p-1}, z_p$ are regarded as known, then x can be re-expressed as below:

$$\begin{aligned} x_j &= b_{jI} Z_I + b_{jII} Z_{II} + b_{jIII} Z_{III} + \dots + b_{ju1} Z_{u1} + b_{ju2} Z_{u2} + b_{ju3} Z_{u3} + \dots \\ &= \sum b_{jp} Z_p \end{aligned} \quad (1)$$

$$\text{Or, } X = BZ + B_u Z_u, \text{ in matrix form.} \quad (2)$$

Let's assume that factor dimensions Z_I, Z_{II}, Z_{III} have been maximized, and then other factor dimensions $Z_{u1}, Z_{u2}, Z_{u3}, \dots$ will be minimized by the eigenvalue-eigenvector procedure, due to the trace of R_{pca} is a consistent. Consequently early eigenvalues are biased upward and later eigenvalues are biased downward. Since the minimized $Z_{u1}, Z_{u2}, Z_{u3}, \dots$ are often similar in magnitude and their combined quadratic distributions represent the shape of a hyper-ball, it is logical to regard them as error

variances extracted from the original data set. As a result of such logical progression, equations (1) and (2) become:

$$x_j = b_{jI}Z_I + b_{jII}Z_{II} + b_{jIII}Z_{III} + \varepsilon_j$$

$$= \sum b_{jm}Z_m + \varepsilon_j \quad (3)$$

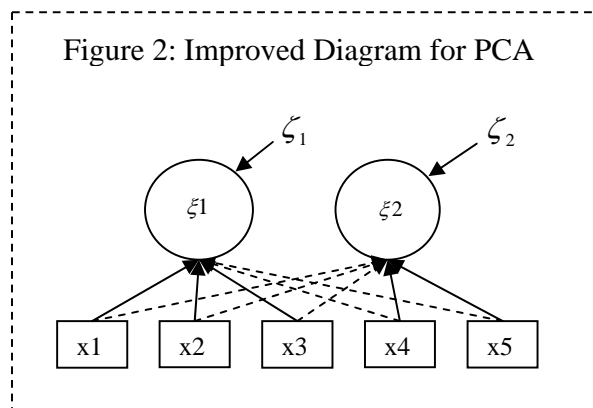
(Where, $\varepsilon_j = b_{ju1}Z_{u1} + b_{ju2}Z_{u2} + b_{ju3}Z_{u3} + \dots$)

Or, $X = BZ + E$, in matrix form. (4)
 (where, $E = B_u Z_u$)

Despite the initial assumption that the data contains no error, the eigenvalue-eigenvector procedure still finds some factor dimensions are very close to error distribution, providing evidence for dimension deletion. As a result, PCA deletes these error dimensions and reduces the full set of p derived factors to m with maximum possible variance. Consequently, observed variables x_1, x_2, \dots, x_p are replaced by m derived variables, z_1, z_2, \dots, z_m , with minimized loss of information (variance). In literature, statements regarding components as simply linear combinations of observables and therefore observables in their own right are only applicable to the full set of z_1, z_2, \dots, z_p .

Equations (3) and (4) demonstrate that by deleting the error component dimensions, PCA has reduced a portion of the unique variance in the original data it begins with. The claim by literature that PCA does not differentiate between common, specific and error variances is essentially incorrect. By simply judging from the matrix R_{pca} PCA use and then verbally reaching a conclusion is a faulty process for quality judgment. In fact, the factor solutions from PCA are much more trustworthy than literature suggested. Draping null components and getting rid of some unique variance in the original data will lend support to the use of PCA. It should be made clear also that the retained components from PCA still contain some unique variance.

The second error in the traditional interpretation of PCA is the diagram presenting the relationship between components and observed variables in Figure 1.a. PCA diagram shown in Figure 1.a is incomplete because it has not counted all variance in observed variables. In comparison, the diagram in Figure 1.b counts 100% variances of all observed variables. Consequently, an improved diagram for PCA is suggested in Figure 2 (below). This new diagram incorporates two disturbance factors to compensate for the variances which are not accounted for by the relationship between components and observed variables.



3.2 Factor Analysis (FA):

In PCA the initial assumption that all variance is systematic (unity diagonal) is certainly false in any given situation. In FA, the total variance of any observed variable will be partitioned into the three terms:

$$\sigma_x^2 = \text{variance due to measurement error} + \text{specific variance} + \text{common variance} \quad (5)$$

According to Nunnally and Bernstein (1994), and Lattin et al (2003), factor analysis makes no effort to separate specific variance from error variance. Consequently, they are combined together as unique variance (u^2). So that the variance equation becomes:

$$\sigma_x^2 = h^2 + u^2 \quad (6)$$

Here, h^2 denotes common variance or commonality; u^2 is also called residual variance.

FA only uses h^2 part of the variance for factor analysis that is what different from PCA. We have to be very careful here to claim that FA is better than PCA, because good intention may not produce good results just like many things in the real world. FA postulate that underlying the observed variables x_1, x_2, \dots, x_p are m unobservable variables, or common factors, y_1, y_2, \dots, y_m . In behavioral science, a deeply held hypothesis is that each observed variable (x_1, x_2, \dots, x_p) is assumed to be linear combinations of underlying factors (y_1, y_2, \dots, y_m), plus an error term. Thus we have:

$$x_j = \sum_{k=1}^m \lambda_{jk} y_k + e_j \quad \text{where } j= 1, 2, \dots, p. \quad (7)$$

$$\text{Or, } X = \Lambda Y + E, \text{ in matrix form.} \quad (8)$$

To fit the hypothesis mentioned above, FA uses symmetric matrix R_{fa} where the 1's on the diagonal of R are replaced by commonalities ($h^2 = 1 - \Psi_j < 1$, and $\Psi_j = \text{var}(e_j)$) of observed variables. As the result of this change, the eigenvalues of R_{fa} can no longer be guaranteed to be positive and therefore, the matrix R_{fa} is indefinite. This is the fundamental difference between FA and PCA. In other words, in FA more factors do not mean more variances explained.

First, in spite of their initial differences in assumption about the variance used in the matrix (R_{fa} vs. R_{pca}), we can see that there is a strong similarity between equations (7), (8) and equations (3), (4).

Second, it is more likely that factor dimensions retained from FA will over-representing common factor variances which are given by h^2 's. One way to illustrate the behavior of different models of EFA is to analyze residual matrices after factors extracted. To make it clear let's go through the following mathematical workout first. Spectral Theorem allows a real symmetric matrix R to be written as its projection form:

$$\begin{aligned}
R = QDQ^T &= [q_1 \quad \cdots \quad q_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \\
&= [\lambda_1 q_1 \quad \cdots \quad \lambda_n q_n] \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \\
&= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T
\end{aligned} \tag{9}$$

Here, each of the terms $\lambda_i q_i q_i^T$ is rank 1 matrix and $q_i q_i^T$ is actually the matrix of the projection onto the subspace spanned by q_i .

In EFA, when no factor has been extracted, the residual matrix $R_0 = R$ (see Eq. (9)). After the first factor is extracted, the residual matrix will be $R_I = \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$ and after the second factor is extracted, the residual matrix will be $R_{II} = \lambda_3 q_3 q_3^T + \cdots + \lambda_n q_n q_n^T$ until $R_{n-1} = \lambda_n q_n q_n^T$. This process may be repeated until all factors have been extracted. For PCA, The values of the successive matrices will shrink toward zero as more of the variance is explained, because all eigenvalues ($\lambda_1 \cdots \lambda_n$) are positive. For FA, the residual values could be negative after first few eigenvectors are extracted out, due to negative λ_i . This means that the first few extracted eigenvectors have supplied additional variances which are not in the matrix R_{fa} .

Third, since the squared multiple correlation in predicting a given variable from the factors defines the communality (h^2) of that variable in FA, unique variance (u^2) becomes in effect the residual not explained by obtained factors. Consequently, the estimated common variance in a common factor model may not contain some common factor variance that has not been included in the model if some of factors have not been extracted (Nunnally and Bernstein, 1994). In other words, FA often underestimates commonalities (h^2) of observed variables and overestimate unique variance (u^2). Unfortunately, there is no foolproof way to distinguish common variance from unique variance. By contrast, PCA is often overestimate commonality by allowing some unique variance of each variable to be included in the factors.

4. CONCLUSION

Historically, FA was regarded as superior over PCA. The judgments about superiority are often based on the intuitive appeal of the matrices (R_{fa} vs. R_{pca}) used for eigenvalue-eigenvector analysis. Since R_{fa} assumes to contain only commonalities of observed variables, FA was recommended. The above mathematical treatments show that both PCA and FA have advantages and disadvantages of their own. PCA solutions contain the all common factor variances plus a portion of unique variances, while that of FA may contain common variances which are not of common factors. As a result, they both may influence the extraction of initial factor differently, but may or may not have influence on final factor (rotated) identified. If both models are used to find the rotated factors simultaneously and the factor solutions of both models are similar, it is more likely to find trustworthy factors in empirical studies. If the factor solutions of different models are different, it is very likely that either the common factor structures are weak or there is too much error contained in the data, or both. Therefore, using both models simultaneously in empirical studies is recommended.

As Nunnally and Bernstein (1994) points out, in empirical studies the two approaches will lead to very similar substantive outcomes when either (1) the number of variables is large; (2) each variable correlated highly with at least some of the remaining variables; and (3) the error variances in the data are reasonably small. Consequently, if the estimates are highly similar under both procedures, you can feel more confident about the result.

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