Multi-period hedging, risk attitude, expected return, and interest rate

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Abstract

This paper develops a multi-period maximum-utility hedging strategy and derives the formula for optimal hedge ratios under this framework by incorporating the impacts of the risk attitude of individuals, expected return of the futures prices, and interest rate into account. We also show that the formula for the multi-period hedging ratio reduces to that of the single-period hedging ratio when the serial correlation in spot and futures price changes is absent and the second moment of the two price changes possess GARCH effects. The empirical evidences show that the out-of-sample hedging performance of the maximum-utility hedging strategy is superior than that of the competing minimum-variance hedging method for both the S&P-500 and FTSE-100 markets, no matter what risk attitudes the individual has.

Key words: Futures markets; multi-period hedging; risk attitude; interest rate

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1 Introduction

Futures markets allow portfolio managers to hedge their risk exposures by shorting stock index futures contracts. The critical problem for hedging centers on the determination of the hedge ratio, i.e., the number of futures contracts to sell for each unit of the underlying asset on which the short hedger bears the price risk. Based on the results in Ederington (1979) and Figlewski (1984), the minimum-variance hedge ratio, b, is equivalent to the ratio of the unconditional covariance between spot and futures price changes over the variance of futures price changes:

$$b = \frac{\mathrm{Cov}(\Delta \mathcal{S}_t, \Delta F_t)}{\mathrm{Var}(\Delta F_t)},$$

where $\Delta \mathcal{S}_t$ and ΔF_t denote the time t price changes in spot and futures prices, respectively. Clearly, the expected return of futures price changes, risk attitude of individuals, and interest rate do not play any role in the minimum-variance hedge ratio.

To incorporate the expected return of futures price changes and risk attitude of individuals into the hedging analysis, Kroner and Sultan (1993) propose a single-period utility-maximizing hedge ratio when the distribution of spot and futures prices is time varying. The time-varying utility-maximizing hedge ratio at time t is given by:

$$b_t = \frac{-E_t\left(\Delta F_{t+1}\right) + 2\gamma \mathrm{Cov}(\Delta \mathcal{S}_{t+1}, \Delta F_{t+1})}{2\gamma \mathrm{Var}(\Delta F_{t+1})},$$

where γ is the degree of risk aversion ($\gamma > 0$) of individuals and $E_t\left(\Delta F_{t+1}\right)$ represents the expected return from the time $t+1$ futures price changes based on the information set at time t. This hedge ratio is more flexible since it not only changes through time as new information arrives, but also varies across individuals with different risk attitudes. Nevertheless, the utility-maximizing hedge ratio proposed by Kroner and Sultan (1993) is only applicable to the one-period hedging analysis.

In practice, hedgers usually require to hedge their risk exposures through a multi-period horizon. In contrast to single-period hedges, the dynamic T-period hedging strategy includes a sequence of hedge ratios, $\{b_0, b_1, \cdots, b_{T-1}\}$. Howard and D'Antonio (1991) propose a backward procedure to determine the sequence of optimal hedge ratios, and apply this procedure to a hedging model in which the spot price changes are autocorrelated but futures returns are not. Based on the idea of Howard and D'Antonio (1991), Lien and Luo (1994) provide the solution for a two-period hedging strategy when spot and futures prices follow a bivariate generalized autoregressive conditional heteroscedasticity (GARCH) model. Particularly, both the mean processes of spot and futures prices in Lien and Luo (1994) are described by error correction models. Low, Muthuswamy, Sakar, and Terry (2002) further derive the formula for the dynamic hedges when the futures prices obey the cost-of-carry model.

The common feature among the aforementioned hedging studies is that all the dynamic multi-period hedging strategies are developed in terms of variance minimum. Essentially, no studies exist concerning the maximum-utility multi-period hedging strategy when $\Delta\mathcal{S}_t$ and ΔF_t follow time-varying distributions. It motivates us to propose a utility-maximizing hedging strategy for a multi-period hedging horizon. The proposed formulas for the multi-period hedging ratios in this research indicate that the hedge ratios not only depend on the relationship between the spot and futures price changes, but also vary across individuals with different risk attitudes, the interest rate, and expected value of futures price changes. The second contribution of the current paper shows that the formula for the multi-period hedging ratio reduces to that of single-period hedges as long as the serial correlation in spot and futures price changes is absent and the two price changes possess the GARCH effect of Bollerslev (1986). The coverage of GARCH models with serial non-correlation is broad, including the martingale process with the bivariate GARCH setting used in Myers (1991), bivariate Markov-switching GARCH model of Lee and Yoder (2007), and long-memory in volatility model developed in Dark (2007) , to name a few. We evaluate the out-of-sample hedging effectiveness of the proposed utility-maximizing hedging strategy in both the S&P-500 and FTSE-100 markets during the financial crisis of 2007-2009. The empirical evidences show that the maximum-utility hedging method outperforms the minimum-variance hedging strategy in terms of utility maximum for both of the two markets.

The remaining parts of this paper are arranged as follows. Section 2 introduces the utility-maximizing multi-period hedging model and develops the solution for the dynamic T-period hedging strategy. Section 3 empirically demonstrates that the hedging performance of the proposed utility-maximizing strategy is superior than that of the competing minimum-variance hedging strategy for both of the S&P-500 and FTSE-100 markets. Section 4 provides a conclusion.

2 The theoretical framework

Suppose that an investor holds a spot position at current time 0 and plans to hedge its risk exposure in the corresponding futures market through time T. Denote the spot and futures price at time t as S_t and F_t, respectively. Both of the price sequences, $\{S_1, S_2, \cdots, S_T\}$ and $\{F_1, F_2, \cdots, F_T\}$, are generated by exogenous stochastic processes.

To offset the risk arose from the price change in the spot position, the hedger shorts a specific number of futures contracts, i.e., b_t, for each unit of the spot asset at each time t. The T-period hedging strategy thus involves a sequence of optimal hedging ratios, $\{b_0, b_1, \cdots, b_{T-1}\}$, in which b_t denotes the number of futures contracts sold at time t in order to hedge the change in the spot price at time $t+1$, i.e., ΔS_{t+1}. Assume that the cash generated from both the spot and futures positions in each period is reinvested at the risk-free rate i and held until the end of period T. Ignoring transaction costs, the end-of-period wealth, W_T, can be presented as:

$$W_T = W_0 + \sum_{t=0}^{T-1} \left[\Delta S_{t+1} - b_t \Delta F_{t+1}\right] (1+i)^{T-t-1}, \tag{1}$$

where i denotes the interest rate, and ΔS_{t+1} and ΔF_{t+1} are the time $t+1$ changes in the spot and futures prices, respectively. We note that no existing studies investigate the impacts of the interest rate on the optimal hedging ratio. However, based on the calculation of W_T specified in Equation (1), this study first introduces the interest rate i into the hedging analysis. Under the setup of $i = 0$, the end-of-period wealth W_T reduces to that in Kroner and Sultan (1993), Lien and Luo (1994), Lien and Wilson (2001), and Low et al. (2002).

To incorporate the expected return of futures prices and the risk attitude into the hedging analysis, we assume that the hedger faces a mean-variance expected utility function:

$$EU_0\left(W_T\right) = E_0\left(W_T\right) - \gamma Var_0\left(W_T\right), \tag{2}$$

where γ is the degree of risk aversion ($\gamma > 0$), and $E_0\left(W_T\right)$ and $Var_0\left(W_T\right)$ denote the expected value and variance of W_T based on the information set at time 0. The mean-variance expected utility function is widely used in the literature to analyze the economic benefits of hedging strategies, including Kroner and Sultan (1993), Lafuente and Novales (2003), Alizadeh and Nomikos (2004), and Alizadeh et al. (2008), to name a few. The sequence of optimal hedging ratios, $\{b_0, b_1, \cdots, b_{T-1}\}$ can be solved by maximizing the expected utility in (2). Given the objective function, this study follows the approach proposed in Howard and D'Antonio (1991), Lien and Luo (1994), and Low et al. (2002) to determin the optimal dynamic hedge by using backward iteration in the following.

We note that the final period hedging decision is made at time $T-1$. According to Equation (1), the mean and variance for W_T are:

$$E_{T-1}\left(W_T\right) = E_{T-1}\left(\Delta S_T\right) - b_{T-1} E_{T-1}\left(\Delta F_T\right), \tag{3}$$

and

$$Var_{T-1}\left(W_T\right) = Var_{T-1}\left(\Delta S_T\right) + b_{T-1}^2 Var_{T-1}\left(\Delta F_T\right) - 2b_{T-1}Cov_{T-1}\left(\Delta S_T, \Delta F_T\right). \tag{4}$$

Based on the objection function in (2), the hedger solves the optimal one-period holdings of futures by maximizing the following expected utility function,

$$\max_{b_{T-1}} EU_{T-1}\left(W_T\right) = \max_{b_{T-1}} \left\{E_{T-1}\left(\Delta S_T\right) - b_{T-1}E_{T-1}\left(\Delta F_T\right) - \gamma\left[Var_{T-1}\left(W_T\right)\right]\right\}, \tag{5}$$

which is maximized by choosing a hedge ratio of

$$b_{T-1} = \frac{-E_{T-1}\left(\Delta F_T\right) + 2\gamma Cov_{T-1}\left(\Delta S_T, \Delta F_T\right)}{2\gamma Var_{T-1}\left(\Delta F_T\right)}. \tag{6}$$

As pointed out in Howard and D'Antonio (1991), at any time t the hedger chooses the optimal hedge ratio b_t in light of the fact that an optimal hedge position is to be taken in each future period. This implies that the sequence of optimal hedging ratios, $\{b_0, b_1, \cdots, b_{T-1}\}$, can be solved by using the backward induction. Based on the solution of b_{T-1} in (6), we follow the approach of Howard and D'Antonio (1991) and determine the hedger's problem backward from time $T-2$ to time 0 in order. The correspoding dynamic hedging strategy is summarized as follows:

Proposition 1: Suppose that both the spot and futures price changes, $\{S_1, S_2, \cdots, S_T\}$ and $\{F_1, F_2, \cdots, F_T\}$, are generated by exogenous processes. For an investor who holds a spot position and plans to hedge the risk exposure through time T, the utility-maximizing hedging strategy can be determined by a sequence of hedging ratios, $\{b_0, b_1, \cdots, b_{T-1}\}$, where

$$\begin{aligned} b_{T-k} = {} & \frac{-E_{T-k}\left(\Delta F_{T-k+1}\right) + 2\gamma(1+i)^{k-1}Cov_{T-k}\left(\Delta S_{T-k+1}, \Delta F_{T-k+1}\right)}{2\gamma(1+i)^{k-1}Var_{T-k}\left(\Delta F_{T-k+1}\right)} \\ & + \frac{2\gamma\sum_{t=T-k+1}^{T-1}(1+i)^{T-t-1}Cov_{T-k}\left(\Delta S_{t+1} - b_t\Delta F_{t+1}, \Delta F_{T-k+1}\right)}{2\gamma(1+i)^{k-1}Var_{T-k}\left(\Delta F_{T-k+1}\right)}, \qquad \forall\, k = 1, 2, \cdots, T. \end{aligned} \tag{7}$$

Please refer to the Appendix for the details about the proofs.

The maximum-utility dynamic hedging strategy is very general, since it incorporates the interest rate, expected returns of futures prices, and risk attitude of individuals into the hedging ratios. Under the settings of $\gamma = 1$, $i = 0$, and $E_{T-k}(\Delta F_{T-k+1}) = 0$, the formula of b_{T-k} derived in (7) is identical to the minimum-variance dynamic hedging strategy proposed in Equation (2) of Low et al. (2002).

The maximum-utility dynamic strategy is not easy to be performed, since b_t is the function of future hedging ratios, including b_{t+1}, b_{t+2}, $\cdots$, and b_{T-1}. Thus, the closed form express for b_t depends on the data-generating-processes (DGP). Along the line of literature, Howard and D'Antonio (1991) propose the closed form express of the minimum-variance hedging ratio for a T-period hedging horizon when the spot price changes are autocorrelated, whereas Lien and Luo (1994) and Low et al. (2002) provide the solutions under a bivariate GARCH model and cost-of-carry model. These close-form solutions are much more complicated than that of single-period hedging ratio, and not easy to be applied. As shown in the following proposition, the second contribution of this study demonstrates that the formula for the multi-period hedging ratio reduces to that of the single-period hedging ratio as long as the serial correlation in spot and futures price changes is absent and the second moments of the two price changes possess GARCH effects of Bollerslev (1986).

Proposition 2: Suppose that both the spot and futures price changes, $\{S_1, S_2, \cdots, S_T\}$ and $\{F_1, F_2, \cdots, F_T\}$, are generated by exogenous processes. Particularly, the spot and futures returns are serial uncorrelation, and the second moments of the two price change follows the GARCH model of Bollerslev (1986). For an investor who holds a spot position and plans to hedge the risk exposure through time T, the utility-maximizing hedging strategy can be determined by a sequence of hedging ratios, $\{b_0, b_1, \cdots, b_{T-1}\}$, where

$$b_{T-k} = \frac{-E_{T-k}\left(\Delta F_{T-k+1}\right) + 2\gamma(1+i)^{k-1} Cov_{T-k}\left(\Delta S_{T-k+1}, \Delta F_{T-k+1}\right)}{2\gamma(1+i)^{k-1} Var_{T-k}\left(\Delta F_{T-k+1}\right)}, \qquad \forall\, k = 1, 2, \cdots, T. \tag{8}$$

Please refer to the Appendix for the details about the proofs.

Clearly, the formula of multi-period utility-maximizing hedging ratio, b_t, proposed in Proposition 2 does not depend on any future hedging ratios. Moreover, it is identical to the solution of single-period utility-maximizing hedging ratio. Under the set-up of $E_{T-k}\left(\Delta F_{T-k+1}\right) = 0$, $i = 0$, and $\gamma = 1$, the formula of multi-period utility-maximizing hedging ratio in (8) reduces to the solution of single-period minimum-variance hedging ratio used in Ederington (1979), Figlewski (1984), Cecchetti et al. (1988), Park and Switzer (1995), Gagnon and Lypny (1995), and Kavussanos and Nomikos (2000). Since the coverage of GARCH models with serial non-correlation is broad, including the martingale process with the bivariate GARCH setting used in Myers (1991), bivariate Markov-switching GARCH model of Lee and Yoder (2007), and long-memory in volatility model developed in Dark (2007) , to name a few, the multi-period utility-maximizing hedging strategy proposed in Proposition 2 will facilitate the hedgers who have a T-period hedging horizon.

3 Empirical hedging analysis

As shown in many empirical studies, the GARCH model has the advantage of incorporating heteroscedasticity into the estimation procedure and capturing the tendency for volatility clustering in financial and economic data. It has induced many scholars to employ multivariate generalized autoregressive conditional heteroscedasticity (GARCH) models to compute time-varying hedge ratios, including Park and Switzer (1995), Gagnon and Lypny (1995), and Kavussanos and Nomikos (2000). To empirical evaluate the hedging performance of the dynamic maximum-utility hedging strategy proposed in Proposition 2, we adopt the bivariate GARCH model employed in Myers (1991) to model the price changes in the spot and futures prices:

$$\Delta S_t = \alpha_s + \varepsilon_{s,t} \tag{9}$$

$$\Delta F_t = \alpha_f + \varepsilon_{f,t}, \tag{10}$$

where

$$\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{f,t} \end{bmatrix} \mid \Psi_{t-1} \sim N(0, H_t), \tag{11}$$

$$H_t \equiv \begin{bmatrix} h^2_{s,t} & h_{sf,t} \\ h_{sf,t} & h^2_{f,t} \end{bmatrix},$$

and Ψ_{T-1} is the information set at time $t-1$. Herein, the variances of ΔS_t and ΔF_t, i.e., $h^2_{s,t}$ and $h^2_{f,t}$, and covariance of ΔS_t and ΔF_t, i.e., $h_{sf,t}$, are specified by:

$$h^2_{s,t} = c_s + a_s \varepsilon^2_{s,t-1} + b_s h^2_{s,t-1} \tag{12}$$

$$h_{sf,t} = c_{sf} + a_{sf} \varepsilon_{s,t-1} \varepsilon_{f,t-1} + b_{sf} h_{sf,t-1} \tag{13}$$

$$h^2_{f,t} = c_f + a_f \varepsilon^2_{f,t-1} + b_f h^2_{f,t-1} \tag{14}$$

Under the framework of the bivariate GARCH model described in Equations (9)-(14), the multi-period utility-maximizing hedging ratio proposed in Proposition 2 can be explicitly expressed as:

$$b_{T-k} = \frac{-\alpha_f + 2\gamma(1+i)^{k-1} h_{sf,T-k+1|T-k}}{2\gamma(1+i)^{k-1} h^2_{f,T-k+1|T-k}}. \qquad \forall\ k = 1, 2, \cdots, T. \tag{15}$$

Based on the explicitly expression of multi-period utility-maximizing hedging ratio, the following compares the hedging performance of the proposed hedging strategy with that of the minimum-variance hedging strategy for both of the S&P-500 and FTSE-100 stock index futures contracts when the spot and futures price changes follow the bivariate GARCH model. The data comprise daily spot and futures prices, ranging from January 4, 2000 to February 26, 2010 and listed in the Datastream Database. To evaluate the hedging performances of the two competing models during the financial crisis rose in 2007, we start the in-sample estimation on January 4, 2000 and reserve the period of the financial crisis for an out-of-sample comparison. In other words, the data used for the in-sample estimation range from January 4, 2000 to February 26, 2010 (2,412 observations), while the data for the out-of-sample hedging comparison start on June 1, 2007 and end on February 26, 2010 (478 observations). Moreover, ΔS_t and ΔF_t are calculated as the differences in the logarithms of prices multiplied by 100. The summary statistics for spot and futures prices are shown in Table 1.

Table 2 displays the empirical out-of-sample performance of the maximum-utility hedging strategy and minimum-variance hedging strategy. The value of risk attitude, γ, is set to be in line with most empirical studies, in which Chou (1988) estimates γ to be 4.5, Poterba and Summers (1986) estimate γ to be 3.5, Grossman and Shiller (1981) find the value of γ to be 4, and Friend and Hasbrouck (1982) demonstrate it to be 6. Accordingly, we assume the risk attitude γ as 2, 3, 3.5, 4, 4.5, 5, and 6 for the out-of-sample hedging comparison. As shown in Table 2, the out-of-sample the maximum-utility hedging strategy produces better utilities on an ex ante basis than the competing minimum-variance hedging method for both S&P-500 and FTSE-100 markets, no matter what risk attitudes the individual has. The results indicate that the maximum-utility hedging strategy outperforms the minimum-variance hedging method during the financial crisis of 2007.

4 Conclusion

This paper proposes a multi-period maximum-utility hedging strategy and provides explicitly solutions for the sequence of optimal hedging ratios when the spot and futures price changes are serial uncorrelated and the second moments of the two price changes possess GARCH effects of Bollerslev (1986). The proposed dynamic hedging strategy allows us to incorporate the risk attitude of individuals, expected return of the hedged portfolio, and interest rate into the hedging analysis. Furthermore, we demonstrate that the formula of the multi-period hedging ratio reduces to that of the single-period hedging ratio, and is very easy to be performed.

We also apply the multi-period maximum-utility hedging strategy to the S&P-500 and FTSE-100 markets. The results indicate that the out-of-sample hedging performances of the proposed strategy is superior to that of the minimum-variance strategy in terms of utility maximum. Since the coverage of GARCH models with serial uncorrelated is broad, the proposed multi-period utility-maximizing hedging strategy will facilitate the hedgers who plan to hedge the changes in spot prices through a T-period horizon.

5 Appendix

Proof of Proposition 1: Consider the hedger's problem at time $T-k$, where $k = 2, 3, \cdots, T$. By using backward induction, the formulas for the sequence of $\{b_{T-1}, b_{T-2}, b_{T-k+1}\}$ are all known, although these hedge ratios may depend on price information after $T-k$. These hedge ratios have to be regarded as stochastic variables at time $T-k$.

Based on Equation (1), the expected value and variance for the end-of-period wealth, W_T, based on time $T-k$ information set are: is given by:

$$\begin{aligned} E_{T-k}\left(W_T\right) &= (1+i)^{k-1} E_{T-k}\left(\Delta S_{T-k+1}\right) - b_{T-k}(1+i)^{k-1} E_{T-k}\left(\Delta F_{T-k+1}\right) \\ &\quad + E_{T-k}\left\{\sum_{t=T-k+1}^{T-1}\left[\Delta S_{t+1} - b_t \Delta F_{t+1}\right](1+i)^{T-t-1}\right\} \end{aligned} \tag{A1}$$

and

$$
\begin{aligned}
Var_{T-k}\left(W_T\right) = {} & Var_{T-k}\left\{\Delta S_{T-k+1}(1+i)^{k-1} + \sum_{t=T-k+1}^{T-1}\left[\Delta S_{t+1} - b_t \Delta F_{t+1}\right](1+i)^{T-t-1}\right\} \\
& + b_{T-k}^2 (1+i)^{2(k-1)} Var_{T-k}\left(\Delta F_{T-k+1}\right) \\
& - 2b_{T-k}(1+i)^{2(k-1)} Cov_{T-k}\left\{\Delta S_{T-k+1}, \Delta F_{T-k+1}\right\} \\
& - 2b_{T-k}(1+i)^{k-1} Cov_{T-k}\left\{\sum_{t=T-k+1}^{T-1}\left[\Delta S_{t+1} - b_j \Delta F_{t+1}\right](1+i)^{T-t-1}, \Delta F_{T-k+1}\right\}.
\end{aligned}
\tag{A2}
$$

By maximizing the objection function in (2), the hedger solves the optimal k-period holdings of futures as:

$$
\begin{aligned}
b_{T-k} = {} & \frac{-E_{T-k}\left(\Delta F_{T-k+1}\right) + 2\gamma(1+i)^{k-1} Cov_{T-k}\left(\Delta S_{T-k+1}, \Delta F_{T-k+1}\right)}{2\gamma(1+i)^{k-1} Var_{T-k}\left(\Delta F_{T-k+1}\right)} \\
& + \frac{2\gamma \sum_{t=T-k+1}^{T-1}(1+i)^{T-t-1} Cov_{T-k}\left(\Delta S_{t+1} - b_t \Delta F_{t+1}, \Delta F_{T-k+1}\right)}{2\gamma(1+i)^{k-1} Var_{T-k}\left(\Delta F_{T-k+1}\right)}.
\end{aligned}
\tag{A3}
$$

Proof of Proposition 2: Suppose that the spot and futures returns are serial uncorrelated, and the second moments of the two price changes follow the GARCH model of Bollerslev (1986). Based on Bohrnstedt and Goldberger (1969), it follows that

$$
\begin{cases}
Cov_t\left(\varepsilon_{s,j}, \varepsilon_{f,j}^2\right) = 0 & \\
Cov_t\left(\varepsilon_{f,j}, \varepsilon_{f,j}^2\right) = 0 & \text{where} \quad t+1 \le j \le T \\
Cov_t\left(\varepsilon_{s,j}, \varepsilon_{s,j}\varepsilon_{f,j}\right) = 0 &
\end{cases}
\tag{$A4$}
$$

According to Proposition 1, the optimal hedge ratio at time $T-2$ is:

$$
b_{T-2} = \frac{-E_{T-2}\left(\Delta F_{T-1}\right) + 2\gamma(1+i) Cov_{T-2}\left(\Delta S_{T-1}, \Delta F_{T-1}\right) + 2\gamma Cov_{T-2}\left(\Delta S_T - b_{T-1}\Delta F_T, \Delta F_{T-1}\right)}{2\gamma Var_{T-1}\left(\Delta F_T\right)}.
$$

Under the assumption that the spot and futures returns changes are serial uncorrelated, we have:

$$
\begin{cases}
Cov_{T-g}\left(\Delta S_u, \Delta S_v\right) = 0 & \\
Cov_{T-g}\left(\Delta S_u, \Delta F_v\right) = 0 & \text{when} \quad u \neq v \\
Cov_{T-g}\left(\Delta F_u, \Delta F_v\right) = 0 &
\end{cases}
\tag{$A5$}
$$

Also, by Bohrnstedt and Goldberger (1969), the b_{T-2} can be represented:

$$Cov_{T-2}\left(b_{T-1}\Delta F_T, \Delta F_{T-1}\right) = E_{T-2}\left(b_{T-1}\right) Cov_{T-2}\left(\Delta F_T, \Delta F_{T-1}\right) + E_{T-2}\left(\Delta F_T\right) Cov_{T-2}\left(b_{T-1}, \Delta F_{T-1}\right)$$
$$+ E_{T-2}\left[\delta_{T-2}\left(\Delta F_{T-1}\right)\delta_{T-2}\left(b_{T-1}\right)\delta_{T-2}\left(\Delta F_T\right)\right], \qquad \text{(A6)}$$

where $\delta_{T-2}(X) = X - E_{T-2}(X)$ for any given random variable X. By Equation (A5),

$$Cov_{T-2}\left(\Delta F_T, \Delta F_{T-1}\right) = 0.$$

We thus note that the first term in Equation (A6) is trivial.

Based on the results in Equation (A4), the part of the second term in Equation (A6), i.e., $Cov_{T-2}\left(\Delta F_{T-1}, b_{T-1}\right)$ can be represented as:

$$\begin{aligned}
&Cov_{T-2}\left(\Delta F_{T-1}, b_{T-1}\right)\\
=&Cov_{T-2}\left(\Delta F_{T-1}, \frac{-E_{T-1}\left(\Delta F_T\right) + 2\gamma Cov_{T-1}\left(\Delta S_T, \Delta F_T\right)}{2\gamma Var_{T-1}\left(\Delta F_T\right)}\right)\\
=&E_{T-2}\left[\frac{-\Delta F_{T-1}E_{T-1}\left(\Delta F_T\right) + 2\gamma\Delta F_{T-1}\left[E_{T-1}\left(\Delta S_T\Delta F_T\right) - E_{T-1}\left(\Delta S_T\right)E_{T-1}\left(\Delta F_T\right)\right]}{2\gamma Var_{T-1}\left(\Delta F_T\right)}\right]\\
-&E_{T-2}\left(\Delta F_{T-1}\right)E_{T-2}\left[\frac{-E_{T-1}\left(\Delta F_T\right) + 2\gamma\left[E_{T-1}\left(\Delta S_T\Delta F_T\right) - E_{T-1}\left(\Delta S_T\right)E_{T-1}\left(\Delta F_T\right)\right]}{2\gamma Var_{T-1}\left(\Delta F_T\right)}\right]
\end{aligned}$$

where

$$\begin{aligned}
&E_{T-2}\left[-\Delta F_{T-1}E_{T-1}\left(\Delta F_T\right) + 2\gamma\Delta F_{T-1}\left[E_{T-1}\left(\Delta S_T\Delta F_T\right) - E_{T-1}\left(\Delta S_T\right)E_{T-1}\left(\Delta F_T\right)\right]\right]\\
=&-E_{T-2}\left(\Delta F_{T-1}\right)E_{T-2}\left(\Delta F_T\right) + 2\gamma E_{T-2}\left(\Delta F_{T-1}\right)\left[E_{T-2}\left(\Delta S_T\Delta F_T\right) - E_{T-2}\left[E_{T-1}\left(\Delta S_T\right)E_{T-1}\left(\Delta F_T\right)\right]\right]
\end{aligned}$$

and

$$\begin{aligned}
&E_{T-2}\left[-E_{T-1}\left(\Delta F_T\right) + 2\gamma\left[E_{T-1}\left(\Delta S_T\Delta F_T\right) - E_{T-1}\left(\Delta S_T\right)E_{T-1}\left(\Delta F_T\right)\right]\right]\\
=&-E_{T-2}\left(\Delta F_T\right) + 2\gamma E_{T-2}\left(\Delta S_T\Delta F_T\right) - 2\gamma E_{T-2}\left[E_{T-1}\left(\Delta S_T\right)E_{T-1}\left(\Delta F_T\right)\right].
\end{aligned}$$

Accordingly, we obtain:

$$Cov_{T-2}\left(\Delta F_{T-1}, b_{T-1}\right) = 0$$

Moreover, the last term in Equation (A6) can be written as:

$$\begin{aligned}
&E_{T-2}\left[\delta_{T-2}\left(\Delta F_{T-1}\right)\delta_{T-2}\left(b_{T-1}\right)\delta_{T-2}\left(\Delta F_{T}\right)\right]\\
=&E_{T-2}\left[\delta_{T-2}\left(\Delta F_{T-1}\right)\delta_{T-2}\left(\Delta F_{T}\right)b_{T-1}\right]-E_{T-2}\left[\delta_{T-2}\left(\Delta F_{T-1}\right)\delta_{T-2}\left(\Delta F_{T}\right)\delta_{T-2}\left(b_{T-1}\right)\right]\\
=&E_{T-2}\left[\delta_{T-2}\left(\Delta F_{T-1}\right)\delta_{T-2}\left(\Delta F_{T}\right)\right]E_{T-2}\left(b_{T-1}\right)-E_{T-2}\left[\delta_{T-2}\left(\Delta F_{T-1}\right)\delta_{T-2}\left(\Delta F_{T}\right)\right]E_{T-2}\left(b_{T-1}\right).
\end{aligned}$$

Since ΔF_T is uncorrelated with ΔF_{T-1}, the last term in Equation (A6) is also trivial. Accordingly, we obtain:

$$Cov_{T-2}\left(b_{T-1}\Delta F_T, \Delta F_{T-1}\right)=0$$

and

$$b_{T-2}=\frac{-E_{T-2}\left(\Delta F_{T-1}\right)+2\gamma(1+i)Cov_{T-2}\left(\Delta S_{T-1}, \Delta F_{T-1}\right)}{2\gamma Var_{T-1}\left(\Delta F_{T}\right)}. \qquad (A7)$$

Similarly, at time $t-k$ we have

$$b_{T-k}=\frac{-E_{T-k}\left(\Delta F_{T-k+1}\right)+2\gamma(1+i)^{k-1}Cov_{T-k}\left(\Delta S_{T-k+1}, \Delta F_{T-k+1}\right)}{2\gamma(1+i)^{k-1}Var_{T-k}\left(\Delta F_{T-k+1}\right)}, \qquad \forall\, k=1,2,\cdots,T. \qquad (A8)$$

The proof of Proposition 2 is now complete.

We note that the formula for multi-period hedge ratios is identical to that of single-period hedge ratios. As a consequence, the multi-period hedge ratio will be reduced to the single-period hedge ratio under the assumption of Equations (A4) and (A5).

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Table 1. The Descriptive Statistics of Daily Returns.

	S&P 500		FTSE 100	
	Spot	Futures	Spot	Futures
Mean	-0.000154	-0.000156	-0.000147	-0.000153
St. Dev	0.013826	0.014004	0.013347	0.013425
Skew	0.128029	0.365949	0.039664	0.023224
Kurtosis	11.69307	14.73377	9.819113	9.962144
Cross Corr	0.978209		0.984637	

The data are daily closed prices, ranging from January 4, 2000 to February 26, 2010.

Table 2. The Utility of the S&P-500 and FTSE-100 during the financial crisis (2000.1 – 2009.3)

	S&P-500		FTSE-100	
	Maximum Utility	Minimum Variance	Maximum Utility	Minimum Variance
γ	Utility	Utility	Utility	Utility
2	-0.3246	-0.3247	-0.1503	-0.1505
3	-0.4873	-0.4875	-0.2263	-0.2264
3.5	-0.5687	-0.5688	-0.2642	-0.2644
4	-0.6500	-0.6502	-0.3022	-0.3023
4.5	-0.7314	-0.7316	-0.3401	-0.3403
5	-0.8128	-0.8129	-0.3781	-0.3782
6	-0.9755	-0.9756	-0.4540	-0.4541

The data used for the in-sample estimation range from January 4, 2000 to February 26, 2010 (2,412 observations), while the data for the out-of-sample hedging comparison start on June 1, 2007 and end on February 26, 2010 (478 observations). The hedge ratio for maximum-utility hedging strategy is calculated by (8), and the hedge ratio for minimum variance hedging strategy is computed by: $\mathrm{Cov}(\Delta S_t, \Delta F_t) / \mathrm{Var}(\Delta F_t)$.